Homework 2, due 9/17

Only your **five** best solutions will count towards your grade.

1. Let $C = \partial D(0,2)$ denote the circle of radius 2 around the origin, oriented positively. Compute the following integrals:

   (a)
   $$\int_C \frac{1}{z^2-1}\,dz$$

   (b)
   $$\int_C \frac{e^z}{(z-1)^n}\,dz$$

   for all integers $n \geq 0$.

2. Suppose that $f : \mathbf{C} \to \mathbf{C}$ is holomorphic, and for some $d, C > 0$ we have
   $$|f(z)| < C(1+|z|)^d \qquad \text{for all } z \in \mathbf{C}.$$

   (a) Show that if $d < 1$, then f is constant.

   (b) Show that if $d \leq k$, then f is a polynomial of degree at most k.

3. Let $f : \mathbf{C} \to \mathbf{C}$ be a non-constant holomorphic function. Show that the image $f(\mathbf{C})$ is dense in $\mathbf{C}$.

4. Let $D = D(0,1) \subset \mathbf{C}$ be the open unit disk, and $u : D \to \mathbf{R}$ be harmonic, i.e.
   $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$
   Here x, y are the real and imaginary parts of z, and you may assume that u has continuous second partial derivatives. Show that there is a holomorphic function $f : D \to \mathbf{C}$ such that $u = \mathrm{Re}(f)$. *(Hint: consider what f' would have to be.)*

5. Suppose that $\gamma : [0,1] \to \mathbf{C}$ is a smooth curve parametrizing the boundary $\partial\Omega$ of an open set $\Omega \subset \mathbf{C}$ oriented positively. Show that the area $A(\Omega)$ is given by
   $$A(\Omega) = \frac{1}{2i}\int_\gamma \bar{z}\,dz.$$

6. Prove that if $N > 0$ is an integer and f is holomorphic on $D(0,2)$ with
   $$|f^{(N)}(0)| = N!\,\sup\{|f(z)| \,:\, |z| = 1\},$$
   then $f(z) = cz^N$ for some $c \in \mathbf{C}$.

7. Suppose that $f(z) = \sum_{n\geq 0} c_n z^n$ defines a holomorphic function on $D(0,1)$, such that $f(z) \in \mathbf{R}$ for all $z \in D(0,1) \cap \mathbf{R}$. Show that $c_n \in \mathbf{R}$ for all n.

8. Consider the improper integral

$$I = \lim_{R\to\infty} \int_0^R e^{ix^2}\, dx$$

on the positive real axis. Prove that

$$I = \lim_{R\to\infty} \int_{\gamma_R} e^{iz^2}\, dz,$$

where γ_R is the line segment $\gamma_R(t) = te^{i\theta}$ for any $\theta \in (0, \pi/2)$, with $t \in [0, R]$. Using that $\int_0^\infty e^{-x^2}\, dx = \sqrt{\pi}/2$, deduce that

$$I = \frac{\sqrt{\pi}}{2} e^{\pi i/4}.$$